Chair for Computation in Engineering
Department of Civil, Geo and Environmental Engineering
Technical University of Munich

Space-Time Finite Element Method

Software Lab Project 2018

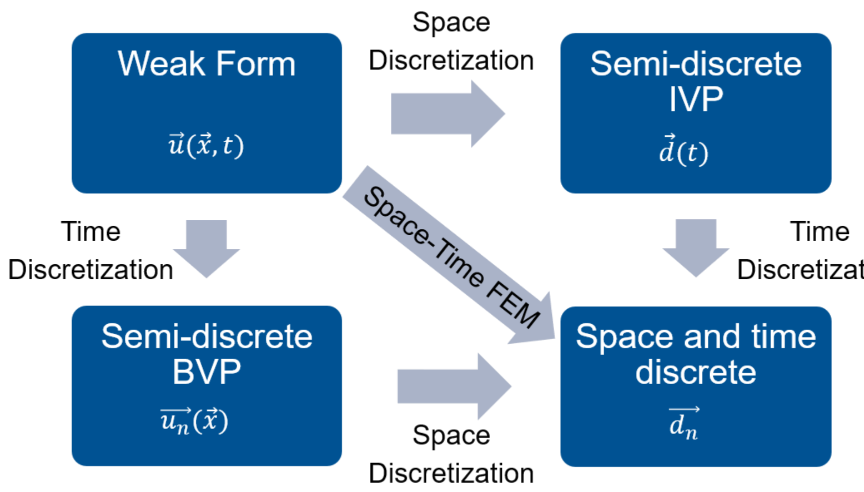
Mahmoud Zidan, Gabriella Loera, Mahmoud Ammar

Mohamed Elhaddad, Philipp Kopp

Abstract

This project shows an implementation of the Space-Time Finite Element Method to solve a transient heat diffusive problem. The implemented program is a stand-alone object-oriented Python code. Also, a convergence plot in the L2 norm with different polynomial orders of the hierarchic shape functions is presented.

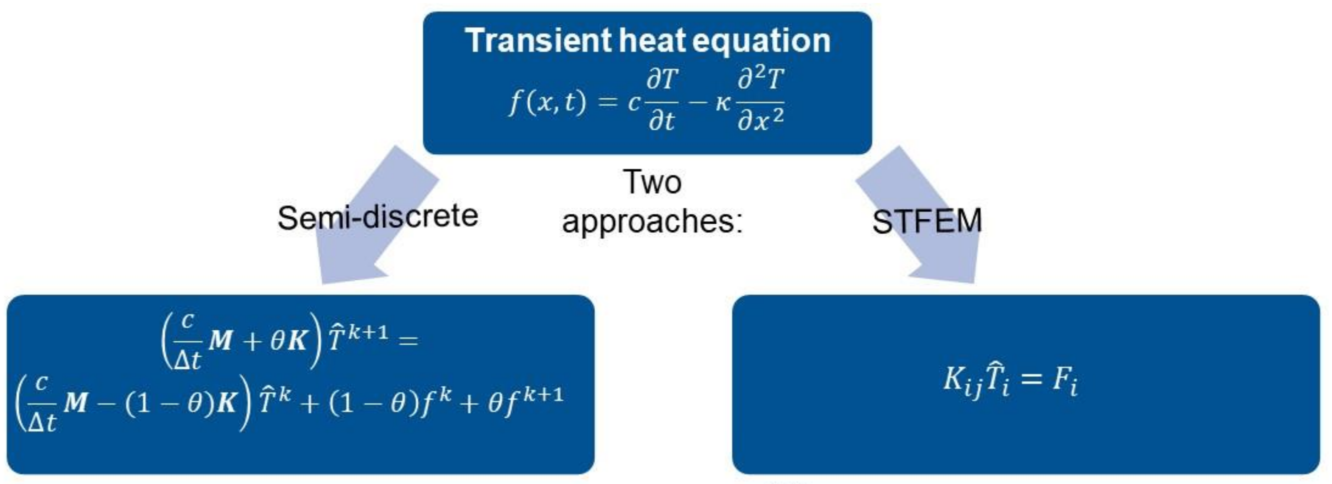
Motivation



Advantages of Space-Time FEM (STFEM):

- Using high-order basis functions for space and time discretization leads to an approximation with a higher accuracy
- Offers the possibility of exploitation of HPC.
- Allows local refinement in space and time.

Theory: semi-discrete vs Space-Time Finite Element Method



Where:

$\hat{T}^k = \sum^{Dof} \boldsymbol{N}\hat{T}_i^k$ ... discretized temperature field

$\boldsymbol{M} = \int \boldsymbol{N}^T\boldsymbol{N}$, $\boldsymbol{K} = \int \boldsymbol{N}_{,x}^T\boldsymbol{N}_{,x}$ — System matrices

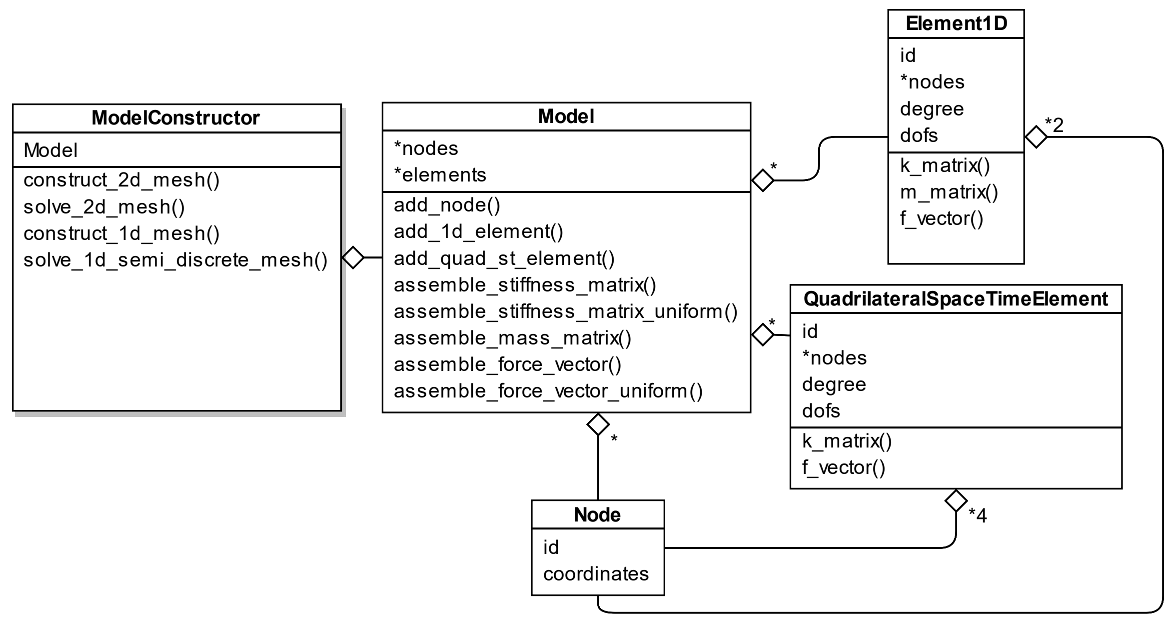
$0 \le \theta \le 1$: $\theta = 0 \rightarrow$ Explicit Euler; $\theta = 1 \rightarrow$ Implicit Euler; $\theta = 0.5 \rightarrow$ Crank-Nicolson

Where:

$$K_{ij} = c\iint \frac{\partial N_i}{\partial t} N_j \, dxdt + \kappa \iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\, dxdt$$

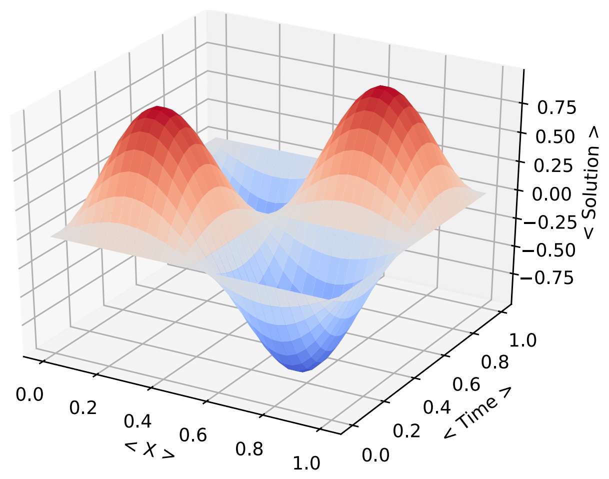
Code Structure

The implemented Python code is done in an object oriented way, where the class "*ModelConstructor*" has the operations to solve the problem in a semi-discrete approach or using space-time FEM. It generates an instance of the class "*Model*" where the assembly of system variables happen.

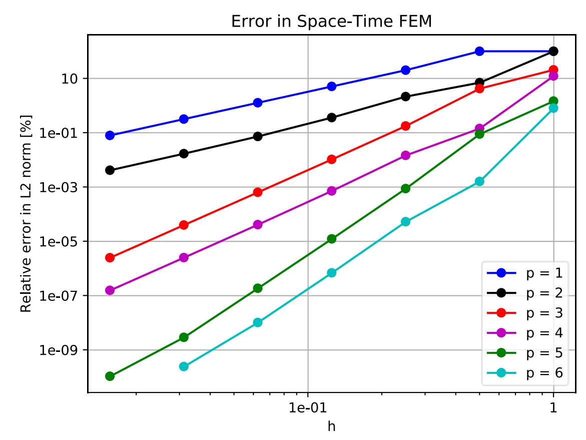


Results obtained using STFEM

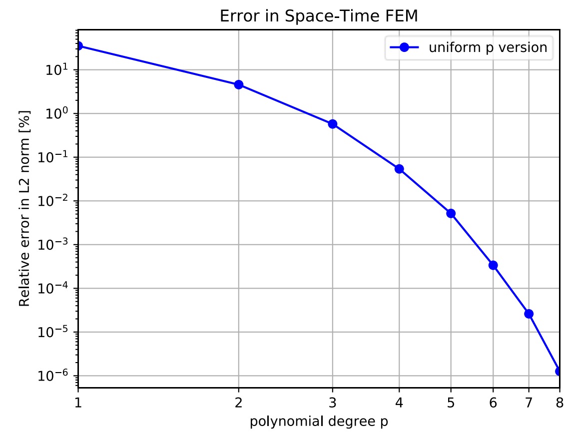
Below, is a result of solution to a heat flux function of $f(x,t) = 2\pi\sin(2\pi x)\cos(2\pi t) + 4\pi^2\sin(2\pi x)\sin(2\pi t)$, with the material constants being equal to one. Also, convergence plots in the L2 norm are shown.



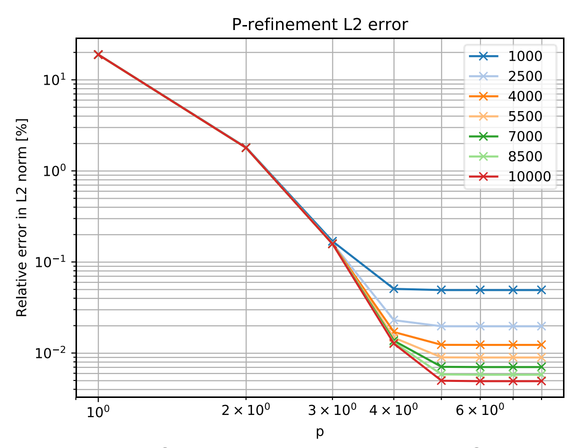
Solution of a transient heat problem using ST FEM approach.



H-refinement L2 error for different polynomial degrees (STFEM)



P-refinement L2 error for a uniform mesh (STFEM)



P-refinement L2 error for a uniform mesh (Semi-Discrete) for a different number of time steps

Conclusions and outlook

A solver for the 1D heat equation using both approaches, the semi-discrete and using space-time FEM, was developed and tested. An improvement to the project would be an extension to higher dimensions, and/or obtaining a partitioned solution for the space-time FEM to reduce the computational effort. Also, an implementation of a non-uniform mesh would allow local time refinement.

References

[1] Thomas J. R. Hughes, Gregory M. Hulbert : *Space-Time Finite Element Methods for Elastodynamics: Formulations and Error Estimates*, Computer Methods in Applied Mechanics and Engineering, 66: 339 – 363, June 1987.

[2] Alexander Düster, *High-Order FEM: Lecture Notes*, Computation in Engineering Chair TUM, April 2011.